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A master function approach for describing shape invariance parameters

H Fakhri[1,2,3] **and S K A Seyed Yagoobi**[2,3]

[1] (IPM) Institute for Studies in Theoretical Physics and Mathematics, Tehran, 19395-5531, Iran
[2] Research Institute for Fundamental Sciences, Tabriz 51664, Iran
[3] Faculty of Physics, Tabriz University, Tabriz 51664, Iran

E-mail: hfakhri@theory.ipm.ac.ir and skasy@tabrizu.ac.ir

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Abstract
With respect to which quantum number they are shape-invariant, most of the one-dimensional shape invariance potentials fall into two classes, for each of which one of the quantum numbers describing the shape invariance is assigned. On this basis, the shape invariance parameters of every class are separately described in terms of a master function, its weight function and an appropriate quantum number.

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There are a large number of one-dimensional quantum mechanical systems whose spectra and wavefunctions are exactly solvable. The most well known of these is the quantum harmonic oscillator [1]. There are other solvable systems with, say, a Morse potential, Scarf potential, Eckart potential, and many others [2]. These solvable potentials have established a close connection with the pioneering work of Infeld and Hull [3] on factorization and the algebraic solution of bound state problems. In fact, some three decades after the work of Infeld and Hull, factorizability and shape invariance were introduced by Witten, Gendenshtein and others [4] as an important aspect of these potentials in the framework of supersymmetric quantum mechanics. Of course, the property of supersymmetry as an approach to solvability as such has so far produced many useful results in physics. Shape invariance symmetry provides the possibility of exactly determining the corresponding wavefunction of the problem, using an algebraic procedure. The corresponding Schrödinger equation for these problems factorizes out into the product of raising and lowering operators, and the energy eigenfunctions of the system transform into each other with the help of these operators [5].

However, it was shown that a wide range of shape invariance potentials lie in two different classes [6, 7]. In the first class, which was obtained from factorization of the Schrödinger equation with respect to the main quantum number, the superpotential was explained in terms of the master function, the corresponding weight function, and also the main quantum number n [6]. The second class was derived from factorizing the Schrödinger equation with respect to the secondary quantum number m, and the superpotential was explained in terms of the master function, its weight function, and also the secondary quantum number m [7]. Such a classification for shape invariance potentials arises from introducing the master function and the corresponding weight function for special functions in mathematical physics. The master function has proved so powerful that even some approaches to solving quantum mechanical problems in two and three dimensions have been introduced [8]. By temporarily neglecting the works of [6] and [7], we see that the problems of shape invariance potentials have so far been speculated and solved by imposing the shape invariance relation on the parameters which also bear the name of shape invariance parameters [9–14]. Now, in this paper, we try to describe these shape invariance parameters for the above-mentioned two superpotential classes in terms of the master function, its corresponding weight function, and also the quantum numbers of each of the two classes. Therefore, it seems most appropriate to review once again the subject of the master function and its approach to these two classes of shape invariance potentials.

In previous papers [7, 15], the master function $A(x)$ was introduced as a polynomial of at most degree two, and a corresponding non-negative weight function $W(x)$ in the same interval (a, b), so that, for a given master function $A(x)$, the weight function is so chosen that the expression $(1/W(x))(\mathrm{d}/\mathrm{d}x)(A(x)W(x))$ is a polynomial of degree at most one, and the end points of the interval (a, b) are so chosen that the expression $A(x)W(x)$, together with all its derivatives, are zero at these points. It has been proven that the polynomials $\Phi_n(x)$ of degree n with a Rodrigues representation

$$\Phi_n(x) = \frac{a_n}{W(x)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (A^n(x)W(x)) \tag{1}$$

are orthogonal with respect to the scalar product defined by the non-negative weight function $W(x)$ in the interval (a, b). The constant a_n is fixed by the normalization condition. These orthogonal polynomials are the well-known special functions of mathematical physics which are obtained from different choices for the master function and its weight functions, e.g. Jacobi polynomials, hypergeometric polynomials, Hermite polynomials, etc. The second-order differential equation in which the polynomials $\Phi_n(x)$ are satisfied is

$$A(x)\Phi_n''(x) + \frac{(A(x)W(x))'}{W(x)}\Phi_n'(x) - \left[n\left(\frac{A(x)W'(x)}{W(x)}\right)' + \frac{n(n+1)}{2}A''(x)\right]\Phi_n(x) = 0 \qquad n = 0, 1, 2, \ldots \tag{2}$$

where the prime means differentiation with respect to x. By differentiating the differential equation (2) m times with respect to x and then multiplying it by $(-1)^m A^{m/2}(x)$, one obtains a new differential equation of order two, called the associated differential equation:

$$A(x)\Phi_{n,m}''(x) + \frac{(A(x)W(x))'}{W(x)}\Phi_{n,m}'(x) + \left[-\frac{1}{2}(n^2+n-m^2)A''(x) + (m-n)\left(\frac{A(x)W'(x)}{W(x)}\right)' - \frac{m^2}{4}\frac{A'^2(x)}{A(x)} - \frac{m}{2}\frac{A'(x)W'(x)}{W(x)}\right]\Phi_{n,m}(x) = 0 \qquad m = 0, 1, 2, \ldots, n. \tag{3}$$

Solutions of the differential equation (3), called the associated special functions related to the master function $A(x)$ and the weight function $W(x)$, have the following Rodrigues representation:

$$\Phi_{n,m}(x) = \frac{a_n(-1)^m}{A^{m/2}(x)W(x)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m} (A^n(x)W(x)). \tag{4}$$

The shape-invariant equations with respect to the parameter n were derived in [6] by factorization performed on differential equation (2) [15]:

$$\begin{aligned} B(n)A(n)\psi_n(x) &= E(n)\psi_n(x) \\ A(n)B(n)\psi_{n-1}(x) &= E(n)\psi_{n-1}(x) \end{aligned} \tag{5}$$

where the factorization spectrum, $E(n)$, is

$$\begin{aligned} E(n) = \frac{n}{4\left[\left(\frac{A(x)W'(x)}{W(x)}\right)' + nA''(x)\right]^2} &\Bigg\{ 4\left(\frac{A(x)W'(x)}{W(x)}\right)'^2 \left(nA'^2(0) - A(0)\left(\frac{A(x)W'(x)}{W(x)}\right)'\right) \\ &- \left(\frac{AW'}{W}\right)(0)\left(A''(x)\left(\frac{AW'}{W}\right)(0) - 2A'(0)\left(\frac{A(x)W'(x)}{W(x)}\right)'\right) \\ &\times\left(2\left(\frac{A(x)W'(x)}{W(x)}\right)' + nA''(x)\right) \\ &+ n^2A''(0)\left(A'^2(0) - 2A''(0)A(0)\right)\left(nA''(0) + 4\left(\frac{A(x)W'(x)}{W(x)}\right)'\right) \\ &- 10nA(0)A''(x)\left(\frac{A(x)W'(x)}{W(x)}\right)'^2 \Bigg\}. \end{aligned} \tag{6}$$

The one-indicial functions $\psi_n(x)$ are obtained as a multiplier of the orthogonal polynomials $\Phi_n(x)$:

$$\psi_n(x) = W^{1/2}(x)\Phi_n(x). \tag{7}$$

The raising and lowering operators of the parameter n, that is, $A(n)$ and $B(n)$ respectively, which are adjoint to each other, have the following explicit form in terms of the master function and the weight function:

$$\begin{aligned} B(n) &= A(x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\left[nA'(x) + \frac{A(x)W'(x)}{W(x)} + n\frac{A'(0)\left(\frac{A(x)W'(x)}{W(x)}\right)' - A''(x)\left(\frac{AW'}{W}\right)(0)}{\left(\frac{A(x)W'(x)}{W(x)}\right)' + nA''(x)}\right] \\ A(n) &= -A(x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\left[nA'(x) + \frac{A(x)W'(x)}{W(x)} + n\frac{A'(0)\left(\frac{A(x)W'(x)}{W(x)}\right)' - A''(x)\left(\frac{AW'}{W}\right)(0)}{\left(\frac{A(x)W'(x)}{W(x)}\right)' + nA''(x)}\right]. \end{aligned} \tag{8}$$

We also recall that in [7] the differential equation (3) was factorized as a product of the raising and lowering operators of the parameter m, from which the shape invariance equations with respect to the parameter m were thus obtained:

$$\begin{aligned} B(m)A(m)\psi_{n,m}(x) &= E(n,m)\psi_{n,m}(x) \\ A(m)B(m)\psi_{n,m-1}(x) &= E(n,m)\psi_{n,m-1}(x) \end{aligned} \tag{9}$$

with the following spectrum:

$$E(n,m) = -(n-m+1)\left[\left(\frac{A(x)W'(x)}{W(x)}\right)' + \frac{1}{2}(n+m)A''(x)\right]. \tag{10}$$

The two-indicial functions $\psi_{n,m}(x)$ are obtained as a multiplier of the associated special functions:

$$\psi_{n,m}(x) = A^{1/4}(x)W^{1/2}(x)\Phi_{n,m}(x). \tag{11}$$

The raising and the lowering operators of the parameter m, that is, $B(m)$ and $A(m)$, which are adjoint to each other, have the following explicit form:

$$\begin{aligned} B(m) &= \sqrt{A(x)}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\frac{A(x)W'(x)}{2W(x)} + \frac{2m-1}{4}A'(x)}{\sqrt{A(x)}} \\ A(m) &= -\sqrt{A(x)}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\frac{A(x)W'(x)}{2W(x)} + \frac{2m-1}{4}A'(x)}{\sqrt{A(x)}}. \end{aligned} \tag{12}$$

We now return to the subject of shape invariance with respect to the parameter n, and conclude the discussion of the one-dimensional quantum solvable models with the shape invariance parameter α_n. For this purpose, while using the following change of variable:

$$\mathrm{d}y = \frac{\mathrm{d}x}{A(x)} \tag{13}$$

in equations (5), and renaming the raising and lowering operators $B(n)$ and $A(n)$ by $A^\dagger(\alpha_n)$ and $A(\alpha_n)$ respectively, we obtain

$$\begin{aligned} A^\dagger(\alpha_n)A(\alpha_n)\psi_n(y) &= E(n)\psi_n(y) \\ A(\alpha_n)A^\dagger(\alpha_n)\psi_{n-1}(y) &= E(n)\psi_{n-1}(y). \end{aligned} \tag{14}$$

The corresponding factorization operators of equation (14) are

$$\begin{aligned} A^\dagger(\alpha_n) &= \frac{\mathrm{d}}{\mathrm{d}y} + W(y,\alpha_n) \\ A(\alpha_n) &= -\frac{\mathrm{d}}{\mathrm{d}y} + W(y,\alpha_n). \end{aligned} \tag{15}$$

We thus see that equations (14) are the Schrödinger equations ($\hbar = 2m = 1$) with the following superpotential and wavefunction:

$$W(y,\alpha_n) = K(n) + \frac{1}{2}\left[nA'(x) + \frac{A(x)W'(x)}{W(x)}\right]_{x=x(y)} \tag{16}$$

$$\psi_n(y) = [W^{1/2}(x)\Phi_n(x)]_{x=x(y)} \tag{17}$$

with

$$K(n) = \frac{n}{2}\frac{A'(0)\left(\frac{A(x)W'(x)}{W(x)}\right)' - A''(x)\left(\frac{AW'}{W}\right)(0)}{\left(\frac{A(x)W'(x)}{W(x)}\right)' + nA''(x)}$$

where the explicit form of the change of variable $x = x(y)$, substituted in equations (16) and (17), is obtained from solving the first-order differential equation (13). Now, as a quantum number in the Schrödinger equations (14), n distinguishes wave mechanical states from each other, and we call it the main quantum number. The Schrödinger equations (14) have the following partner potentials in the one-dimensional y space:

$$\begin{aligned} V_\pm(y,\alpha_n) &= W^2(y,\alpha_n) \pm \frac{\mathrm{d}}{\mathrm{d}y}W(y,\alpha_n) \\ &= \left[\frac{1}{4}\left(nA'(x) + \frac{A(x)W'(x)}{W(x)}\right)^2 + K(n)\left(nA'(x) + \frac{A(x)W'(x)}{W(x)}\right)\right. \\ &\quad \left.\pm\frac{1}{2}\left(nA''(x) + \left(\frac{A(x)W'(x)}{W(x)}\right)'\right)A(x)\right]_{x=x(y)} + K^2(n). \end{aligned} \tag{18}$$

For the partner potentials $V_+(y,\alpha_n)$ and $V_-(y,\alpha_{n+1})$ equations (14) also describe the shape invariance condition in the following form:

$$\begin{aligned} R(\alpha_n) &:= V_+(y,\alpha_n) - V_-(y,\alpha_{n+1}) \\ &= E(n) - E(n+1). \end{aligned} \tag{19}$$

With the help of equation (6) one can calculate the explicit form of equation (19) in terms of the master function, the weight function, and the main quantum number n. On the other hand, if the explicit functionality of $R(\alpha_n)$ is determined from α_n, one can calculate α_n as the shape invariance parameter in terms of the master function, the weight function, and the main quantum number n. Hence, one can consider the operators $A^\dagger(\alpha_n)$, $A(\alpha_n)$ and the superpotential $W(y,\alpha_n)$, and also the partner potentials $V_\pm(y,\alpha_n)$, to depend on the shape invariance parameter α_n through the quantum number n. If, as usual, we relate the subtraction (19) by a linear relation to the shape invariance parameter α_n then, generally speaking, the relation between α_n and n will be nonlinear for any master function $A(x)$. Therefore, the shape invariance parameters α_n and α_{n-1} will be neither related to each other by a shift nor by a scaling. According to the table in [6] these shape invariance models are those for which $A''(x) \neq 0$ and $A'(0)(A(x)W'(x)/W(x))' - A''(0)(AW'/W)(0) \neq 0$. According to equation (6), for the special cases for which

$$A''(x) = 0 \tag{20a}$$

or

$$A'(0)\left(\frac{A(x)W'(x)}{W(x)}\right)' - A''(0)\left(\frac{AW'}{W}\right)(0) = 0 \tag{20b}$$

we get

$$R(\alpha_n) = \begin{cases} A(0)\left(\dfrac{A(x)W'(x)}{W(x)}\right)' - A'(0)\dfrac{AW'}{W}(0) & \\ -(2n+1)A'^2(0) & \text{for} \quad (20a) \\ A(0)\left(\dfrac{A(x)W'(x)}{W(x)}\right)' - \dfrac{1}{2}A'(0)\dfrac{AW'}{W}(0) & \\ -\dfrac{2n+1}{4}(A'^2(0) - 2A''(0)A(0)) & \text{for} \quad (20b) \end{cases} \tag{21}$$

where, of course, for the special case $A(x) = 1$, conditions (20a) and (20b) are equal. It is clear that only in the special cases (20a) and/or (20b) will the relation between $R(\alpha_n)$ and n be linear. Therefore, by defining the linear relation between the shape invariance parameters α_n and $R(\alpha_n)$ in these special cases as [14]

$$\alpha_n = \tfrac{1}{2}(1 + R(\alpha_n)) \tag{22}$$

one can calculate it in terms of the master function $A(x)$, the weight function $W(x)$, and the main quantum number n:

$$\alpha_n = \begin{cases} \dfrac{1}{2}\left[A(0)\left(\dfrac{A(x)W'(x)}{W(x)}\right)' - A'(0)\dfrac{AW'}{W}(0)\right. & \\ \left. -(2n+1)A'^2(0) + 1\right] & \text{for} \quad (20a) \\ \dfrac{1}{2}\left[A(0)\left(\dfrac{A(x)W'(x)}{W(x)}\right)' - \dfrac{1}{2}A'(0)\dfrac{AW'}{W}(0)\right. & \\ \left. -\dfrac{2n+1}{4}\big(A'^2(0) - 2A''(0)A(0)\big) + 1\right] & \text{for} \quad (20b) \end{cases} \tag{23}$$

or

$$\alpha_n = \begin{cases} \alpha_{n-1} - A'^2(0) & \text{for} \quad (20a) \\ \alpha_{n-1} - \frac{1}{4}(A'^2(0) - 2A''(0)A(0)) & \text{for} \quad (20b). \end{cases} \tag{24}$$

Hence, it is clear from the table in [6] that in the case of a shifted oscillator ($A(x) = 1$) the shape invariance parameter α_n is constant, but for the Morse potential ($A(x) = x$) the shape invariance parameter reduces by one unit. Also, for the Rosen–Morse II potential ($A(x) = 1 - x^2$ with weight function $W(x) = (1 - x^2)^\alpha$, $\alpha > -1$) and for the Eckhart potential ($A(x) = x^2 - 1$ with weight function $W(x) = (x^2 - 1)^{2k}$, $k \in \{0, 1, 2, \ldots\}$) the value of the shift for the shape invariance parameter α_n is -1. But, for quantum solvable models corresponding to $A(x) = x(1-x)$ with weight function $W(x) = x^\alpha(1-x)^\alpha$, $\alpha > -1$ and $A(x) = x(1+x)$ with weight function $W(x) = x^{2k}(1+x)^{2k}$, $k \in \{0, 1, 2, \ldots\}$ the value of the shift for the shape invariance parameter is $-1/4$.

For the cases in which $A''(x) \neq 0$ and $A'(0)(A(x)W'(x)/W(x))' - A''(0)(AW'/W)(0) \neq 0$, equation (22) does not lead to a linear relation between α_n and n. This indicates that, in the general case, the master function approach to the solution of the shape invariance potentials (with respect to the main quantum number n) is an approach which is not obtainable with the use of an appropriate definition for shape invariance parameter such that it changes either by a shift or by a scaling. However, these solutions *do* exist, as proven in [6]. Therefore, apart from expressing the shape invariance parameter α_n in terms of the master function $A(x)$, the weight function $W(x)$ and the main quantum number n, in general, the subject of shape invariance holds as ever, in the sense that the result of the expression $V_+(y, \alpha_n) - V_-(y, \alpha_{n+1})$ in equation (19) is certainly an expression which is independent of y.

In the general framework of shape invariance, we pay attention to the point that this class of solvable models is also calculable by the algebraic method although the explicit form of the parameter α_n in terms of the specifications of the potential may not be used. From the shape invariance relation (19) the spectrum of the nth quantum state is obtained in terms of the spectra of the ground state $E(0)$:

$$E(n) = E(0) - \sum_{k=0}^{n-1} R(\alpha_k). \tag{25}$$

The shape invariance equations (14) yield the following relation for the raising and lowering of the one-dimensional wavefunction $\psi_n(y)$ with respect to the parameter n:

$$A^\dagger(\alpha_n)\psi_{n-1}(y) = \sqrt{E(0) - \sum_{k=0}^{n-1} R(\alpha_k)}\psi_n(y) \tag{26a}$$

$$A(\alpha_n)\psi_n(y) = \sqrt{E(0) - \sum_{k=0}^{n-1} R(\alpha_k)}\psi_{n-1}(y). \tag{26b}$$

For $n = 0$, equation (26b) gives a differential equation of first order for the ground state $\psi_0(y)$:

$$A(\alpha_0)\psi_0(y) = 0 \tag{27}$$

whose solution, with the help of equation (13), is

$$\psi_0(y) = a_0[W^{1/2}(x)]_{x=x(y)}. \tag{28}$$

Now, by making use of equation (26a), one can calculate algebraically all other quantum states from the ground state $\psi_0(y)$ as if this had been done by the shape invariance parameters α_n:

$$\psi_n(y) = \frac{A^\dagger(\alpha_n)}{\sqrt{E(0) - \Sigma_{k=0}^{n-1} R(\alpha_k)}} \cdots \frac{A^\dagger(\alpha_2)}{\sqrt{E(0) - \Sigma_{k=0}^{1} R(\alpha_k)}} \frac{A^\dagger(\alpha_1)}{\sqrt{E(0) - R(\alpha_0)}}\psi_0(y). \tag{29}$$

Now we return to the second class of one-dimensional quantum solvable models described by the shape invariance parameter β_m. Imposing the following change of variable on the shape invariance relations (9):

$$\mathrm{d}y = \frac{\mathrm{d}x}{\sqrt{A(x)}} \tag{30}$$

the factorized one-dimensional Schrödinger equations ($\hbar = 2m = 1$) are written in the following form:

$$\begin{aligned} A^{\dagger}(\beta_m)A(\beta_m)\psi_{n,m}(y) &= E(n,m)\psi_{n,m}(y) \\ A(\beta_m)A^{\dagger}(\beta_m)\psi_{n,m-1}(y) &= E(n,m)\psi_{n,m-1}(y) \end{aligned} \tag{31}$$

where the raising and lowering operators $A^{\dagger}(\beta_m)$ and $A(\beta_m)$ are the renamed operators of $B(m)$ and $A(m)$:

$$\begin{aligned} A^{\dagger}(\beta_m) &= \frac{\mathrm{d}}{\mathrm{d}y} + W(y,\beta_m) \\ A(\beta_m) &= -\frac{\mathrm{d}}{\mathrm{d}y} + W(y,\beta_m). \end{aligned} \tag{32}$$

The superpotential $W(y,\beta_m)$ and the wavefunction $\psi_{n,m}(y)$ are thus obtained:

$$W(y,\beta_m) = -\left[\frac{\frac{A(x)W'(x)}{2W(x)} + \frac{2m-1}{4}A'(x)}{\sqrt{A(x)}}\right]_{x=x(y)} \tag{33}$$

$$\psi_{n,m}(y) = \left[A^{1/4}(x)W^{1/2}(x)\Phi_{n,m}(x)\right]_{x=x(y)} \tag{34}$$

where the explicit form of the change of variable $x = x(y)$ is obtained by solving the first-order differential equation (30). We note that, next to n, m is also a number that distinguishes between the quantum states. For this reason, we call m the secondary quantum number. Equations (31) are the Schrödinger equations in one-dimensional space with coordinate y (obtained from equation (30)). Their corresponding partner potentials are respectively

$$\begin{aligned} V_{\pm}(y,\beta_m) &= W^2(y,\beta_m) \pm \frac{\mathrm{d}}{\mathrm{d}y}W(y,\beta_m) \\ &= \left[\frac{1}{4A(x)}\left(\frac{A(x)W'(x)}{W(x)}\right)^2 + \frac{(2m-1)(2m-1\pm 2)}{16}\frac{A'^2(x)}{A(x)}\right. \\ &\quad \left. + \frac{2m-1\pm 1}{4}\frac{A'(x)W'(x)}{W(x)} \mp \frac{1}{2}\left(\frac{A(x)W'(x)}{W(x)}\right)' \mp \frac{2m-1}{4}A''(x)\right]_{x=x(y)}. \end{aligned} \tag{35}$$

By subtracting the shape invariance equations (31), we obtain the shape invariance condition on the partner potentials $V_+(y,\beta_m)$ and $V_-(y,\beta_{m+1})$:

$$\begin{aligned} R(\beta_m) &:= V_+(y,\beta_m) - V_-(y,\beta_{m+1}) \\ &= E(n,m) - E(n,m+1) \\ &= -mA''(x) - \left(\frac{A(x)W'(x)}{W(x)}\right)'. \end{aligned} \tag{36}$$

With the following choice for the functionality of the shape invariance parameter [14]:

$$\beta_m = \tfrac{1}{2}(1 + R(\beta_m)) \tag{37}$$

we obtain β_m in terms of the master function $A(x)$, the weight function $W(x)$, and the secondary quantum number m as

$$\beta_m = \frac{1}{2}\left[-\left(\frac{A(x)W'(x)}{W(x)}\right)' - mA''(x) + 1\right] \tag{38}$$

or

$$\beta_m = \beta_{m-1} - \tfrac{1}{2}A''(x). \tag{39}$$

Now, because of relation (38), one can consider the operators $A^\dagger(\beta_m)$, $A(\beta_m)$, the superpotential $W(y, \beta_m)$, and the partner potentials $V_\pm(y, \beta_m)$ to be functions of β_m through the secondary quantum number m. It is clear that, in this case, the functionality of the shape invariance parameter β_m from the secondary quantum number m is linear. Upon referring to table 1 of [7] one can see that, for the shifted oscillator potentials ($A(x) = 1$) and the three-dimensional oscillator ($A(x) = x$), the shape invariance parameter β_m is constant and does not shift. However, for the Morse potential ($A(x) = x^2$), the Scarf II potential ($A(x) = 1 + x^2$), the generalized Pöschl–Teller potential ($A(x) = x^2 - 1$), and the Natanzon potential ($A(x) = 4x^2 - 1$) the value of the shift for the shape invariance parameter is -1, except for the last case which is -2. For the Scarf I potential ($A(x) = x(1 - x)$) and for $A(x) = 1 - x^2$ the shape invariance parameter shift is +1.

The shape invariance relation (36) expresses the mth quantum state spectrum in terms of the ground state spectrum $E(n, 0)$ by the following relation:

$$\begin{aligned} E(n, m) &= E(n, 0) - \sum_{k=0}^{m-1} R(\beta_k) \\ &= E(n, 0) + m\left(\frac{1}{2}(m-1)A''(x) + \left(\frac{A(x)W'(x)}{W(x)}\right)'\right). \end{aligned} \tag{40}$$

From the shape invariance relations (31) one can obtain the raising and lowering relations of the one-dimensional wavefunction $\psi_{n,m}(y)$ in terms of the parameter m in the following way:

$$A^\dagger(\beta_m)\psi_{n,m-1}(y) = \sqrt{E(n, 0) - \Sigma_{k=0}^{m-1} R(\beta_k)}\psi_{n,m}(y) \tag{41a}$$

$$A(\beta_m)\psi_{n,m}(y) = \sqrt{E(n, 0) - \Sigma_{k=0}^{m-1} R(\beta_k)}\psi_{n,m-1}(y). \tag{41b}$$

For $m = n + 1$, equation (41a) yields

$$A^\dagger(\beta_{n+1})\psi_{n,n}(y) = 0. \tag{42}$$

The highest secondary quantum state, according to the first-order differential equation (42), is given by

$$\psi_{n,n}(y) = a_n(-1)^n[W^{1/2}(x)A^{(2n+1)/4}(x)]_{x=x(y)}. \tag{43}$$

Now, with the help of equation (41b), one can algebraically obtain all the corresponding wavefunctions of the other quantum states using the information embedded in the shape invariance parameters β_m

$$\begin{aligned} \psi_{n,m}(y) = {} & \frac{A(\beta_{m+1})}{\sqrt{E(n, 0) - \Sigma_{k=0}^{m} R(\beta_k)}} \cdots \frac{A(\beta_{n-1})}{\sqrt{E(n, 0) - \Sigma_{k=0}^{n-2} R(\beta_k)}} \\ & \times \frac{A(\beta_n)}{\sqrt{E(n, 0) - \Sigma_{k=0}^{n-1} R(\beta_k)}}\psi_{n,n}(y). \end{aligned} \tag{44}$$

In conclusion, we see that most of the shape invariance potentials are classified into two classes, whether they are shape invariant with respect to the main quantum number n, or to the secondary quantum number m. The property of shape invariance, however, implies the existence of a shape invariance parameter, which is α_n for the first class and β_m for the second one. In each of these distinct cases the shape invariance parameter is expressible in terms of the master function, its weight function, and the quantum number with respect to which the factorization has been done. For the shape invariance potentials corresponding to the main quantum number n, for every master function $A(x)$, the shape invariance parameter α_n does not necessarily vary linearly with n. But for shape invariance potentials corresponding to the secondary quantum number m, for every master function $A(x)$ the shape invariance parameter β_m necessarily varies linearly with m. With regard to the solvable shape invariance models discussed in this paper the parameters α_n and β_m take their role in describing the shape invariance from the quantum numbers n and m, and even the principal role of describing shape invariance is up to the quantum numbers n and m.

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